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Research Article

PROPERTIES AND FUNDAMENTAL THEOREM OF $m-$ PLURISUBHARMONIC FUNCTIONS

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ABSTRACT

In particular, on X there ex ist -plurisubharmonic functions, -convex domains, φ-convex bound aries, etc, all inter-related and having a number of good properties.In this paper we show that, in a strong sense, the plurisubharmonic will be.

KEYWORDS

Plurisubharmonic functions, -convex domains, are subharmonic, maximum principle, unitar matrix, Subharmonic Functions.

INTRODUCTION

In this part, we study the real functions in the space $\square^n$.

Definition 1. the function $u(x)\in L^1_{loc}(D)$ in the domain $D\subset\square^n$ is called $m-$ plurisubharmonic function in D (m sized subharmonic function in a real plain) if $1\le m\le n$

1) is semi-continuous from above in the domain D, that is

$$\overline{\lim_{x\to x_0}} u(x)=\lim_{\varepsilon\to 0}\sup_{B(x^0,\varepsilon)} u(x)\le u(x^0)$$

2) For each $\Pi \subset \mathbb{R}^n$, $\dim_{\mathbb{R}^n} \Pi = m$, it is $u|_{\Pi} \in sh(\Pi \cap D)$.

We mark this class of functions as $m-psh(D)$. For convenience, we include $u \equiv -\infty$ in the class $m-psh(D)$. In the domain $D \subset \mathbb{R}^n$, the functions $m-psh$ are subharmonic in the domain $D \subset \mathbb{R}^n$. In addition to this, the functions $m-psh$ contain all characteristics of the subharmonic functions.

We provide the following characteristics:

1^0. If $u_j(x) \in m-psh(D)$, $a_j \geq 0$, $j = 1,2...,n$, then

$a_1u_1(x) + a_2u_2(x) + ... + a_nu_n(x) \in m-psh(D)$

2^0. If $u_j(x) \in m-psh(D)$, $u_j(x) \geq u_{j+1}(x)$ $j = 1,2...... \Rightarrow$

$\lim_{j\to\infty} u_j(x) \in m-psh(D)$

3^0. If $u_j(x) \in m-psh(D)$, $j = 1,2....$, $u_j(x) \to u(x)$, in this case

$u(x) \in m-psh(D)$

4^0. (maximum principle) If $u(x^0) = \sup_{z\in D} u(x)$ for $u(x) \in m-psh(D)$ and $x^0 \in D$, then $u(x) = const$

.

5^0. If $u(x) \in m-psh(D)$, in this case $u_j(x) = u * K_{1/j}(x-y)$ belongs to $m-psh(D)$, in addition, in $j \to \infty$, it is $u_j(x) \downarrow u(x)$.

Definition 2. If the maximum principle is appropriate for the function $u(x) \in m-psh(D)$ in the domain D, which is from the inequality $\underline{\lim}_{x\to\partial D} (u(x) - v(x)) \geq 0$ it is appropriate the inequality $u(x) \geq v(x)$ for $\forall x \in D$, then the function $u(x)$ is called maximal $m-psh$ function.

There is an equivalent definition of maximal functions, i.e for $u(x) \in m-psh(D)$ to be maximal, in the domain $G \subset\subset D$, for the function $\forall$ $v(x) \in m-psh(D)$

$$u|_{\partial G} \geq v|_{\partial G} \Rightarrow u(x) \geq v(x) \ \ \forall x \in G$$

is necessary and enough.

In conclusion, we look through the matrix $\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right)$ of geometric character of maximal functions $m-psh$, it is a symmetric matrix, after we perform a unitary substitution, it becomes a corresponding diagonal matrix.

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right) \rightarrow \begin{pmatrix} \lambda_1 & 0 & .... & 0 \\ 0 & \lambda_2 & .... & 0 \\ .... & .... & .... & .... \\ 0 & 0 & .... & \lambda_n \end{pmatrix}$$

Here, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) = (\lambda_1(x), \lambda_2(x), ..., \lambda_n(x))$ are the eigenvalues of this matrix.

We prove the criterion by the eigenvalues of the Hessian matrix for $m-$plurisubharmonic functions of class $C^2(D)$.

Theorem 1 For $\forall x \in D$ and all $1 \leq j_1 < j_2 < ... < j_m \leq n$, it is necessary and sufficient for inequalities

$$\lambda_{j_1}(x) + \lambda_{j_2}(x) + ... + \lambda_{j_m}(x) \geq 0$$

to be appropriate in order the function $u(x) \in C^2(D)$ to be $m-$ plurisubharmonic

Proof. (necessity). Let $u \in C^2(D) \cap m-psh(D)$, we provide the solution of inequality

$$\lambda_{j_1} + \lambda_{j_2} + ... + \lambda_{j_m} \geq 0$$

for any $1 \leq j_1 < j_2 < ... < j_m \leq n$.

For this, let any $m-$sized Π plane exist, suppose this plane is given in a system in parametric form

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$$\begin{cases} x_1 = a_{10} + a_{11}t_1 + \ldots + a_{1m}t_m \\ x_2 = a_{20} + a_{21}t_1 + \ldots + a_{2m}t_m \\ \qquad \vdots \\ x_n = a_{n0} + a_{n1}t_1 + \ldots + a_{nm}t_m \end{cases}$$

When $u|_{\Pi} \in sh(\Pi \cap D)$, we determine the Laplacian for the segment function using the calculation of the derivative of the complex function with respect to the plane Π and it forms as:

$$\Delta_{\Pi} u = \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{p=1}^{n} \frac{\partial^2 u}{\partial x_k \partial x_p} a_{kj} a_{pj}$$

Since the segment function is a subharmonic function, this symmetrical form

$$\Delta_{\Pi} u = \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{p=1}^{n} \frac{\partial^2 u}{\partial x_k \partial x_p} a_{kj} a_{pj} \geq 0$$

can be made diagonal by unitary substitution. The matrix of this form is as follows

$$\begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \ldots & 0 & \ldots. & \frac{\partial^2 u}{\partial x_1 \partial x_n} & \ldots & 0 \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & \ldots & \frac{\partial^2 u}{\partial x_1^2} & & 0 & \ldots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \ldots & 0 & \ldots & \frac{\partial^2 u}{\partial x_n^2} & \ldots & 0 \\ \vdots & \ddots & \vdots & \ldots & \vdots & \ddots & \vdots \\ 0 & \ldots & \frac{\partial^2 u}{\partial x_n \partial x_1} & \ldots & 0 & \ldots & \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix}$$

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This matrix is positive definite and if we make it diagonal $\Delta_{\Pi}U=\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{p=1}^{n}\frac{\partial^2 u}{\partial x_k \partial x_p}a_{kj}a_{pj}$ =(if we perform a unitary substitution)=

$$\lambda_1(T^2_{\nu_1^{(1)}}+\ldots+T^2_{\nu_m^{(1)}})+\lambda_2(T^2_{\nu_1^{(2)}}+\ldots+T^2_{\nu_m^{(2)}})+\ldots+\lambda_n(T^2_{\nu_1^{(n)}}+\ldots+T^2_{\nu_m^{(n)}})\geq 0.$$

For the chosen indexes $j_1, j_2,\ldots, j_m$ coefficients of a_{jk} can be chosen in a way that, after unitary substitution , it becomes $T_{\nu_l^{(j)}}=0,\ \forall l=\overline{1,m},\ j\neq j_1, j_2,\ldots, j_m\ (T^2_{\nu_1^{(jk)}}+\ldots+T^2_{\nu_m^{(jk)}})=1$ and as a result

$$\lambda_{j_1}+\lambda_{j_2}+\ldots+\lambda_{j_m}\geq 0$$

Sufficiency. Let's suppose that for $\forall x\in D$ and all $1\leq j_1<j_2<,\ldots,<j_m\leq n$, the inequalities

$$\lambda_{j_1}(x)+\lambda_{j_2}(x)+\ldots+\lambda_{j_m}(x)\geq 0$$

are appropriate. Let us consider this Laplacian by taking an arbitrarily assigned plane Π from $G(m,n)$, a set of m -dimensional planes that are part of the $\mathbb{R}^n$ space:

$$\Delta_{\Pi}U=\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{p=1}^{n}\frac{\partial^2 u}{\partial x_k \partial x_p}a_{kj}a_{pj}$$

$G(m,n)$ is a real multiplicity, which can be viewed as a projective space of $m(n-m)+1$ homogeneous coordinates. In this case, if we take plucker coordinates as local coordinates, for each assigned $x_0\in D$, the $\Delta_{\Pi}U$ operator represents the defined functional in the $G(m,n)$ space. Now, let's look at the plane in vectors

$$A_1=(1,0,\ldots,0,a_{m+11},\ldots,a_{n1})$$

$$A_2=(0,1,\ldots,0,a_{m+12},\ldots,a_{n2})$$

:

$$A_m=(0,\ldots,0,1,a_{m+1m},\ldots,a_{nm})$$

If we calculate the coordinates of Plucker, it is $a_{jk}=(-1)^{k+j}p_{jk}$, $p_{00}=1$, and the rest of the coordinates will be a multiple of these. If these vectors are subjected to a unitary displacement,

$$A_jU=\left(\left(\alpha_{j_1}+\sum_{s=m+1}^{n}a_{js}\alpha_{s,1}\right),\left(\alpha_{j_2}+\sum_{s=m+1}^{n}a_{js}\alpha_{s,2}\right),\ldots,\left(\alpha_{j_n}+\sum_{s=m+1}^{n}a_{js}\alpha_{sn}\right)\right)$$

And our operator gets the form

$$\Delta_\Pi U=\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{p=1}^{n}\frac{\partial^2u}{\partial x_k\partial x_p}a_{kj}a_{pj}=$$

$$=\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_k\left(\alpha_{jk}+\sum_{s=m+1}^{n}a_{sj}\alpha_{sk}\right)^2=\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_k\left(\alpha_{jk}p_{00}+\sum_{s=m+1}^{n}(-1)^{s+j}p_{sj}\alpha_{sk}\right)^2$$

We consider the following homogeneous functional instead

$$\Omega(\Pi)=\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_k\left(\alpha_{jk}p_{00}+\sum_{s=m+1}^{n}(-1)^{s+j}p_{js}\alpha_{sk}\right)^2.$$

Positive definiteness of this functional is equivalent to positive definiteness of $\Delta_\Pi u$. Now we derive the theorem from this lemma

Lemma 1 Let unitar matrix $U=\begin{pmatrix}\alpha_{11}&\ldots&\alpha_{1n}\\ \vdots&\ddots&\vdots\\ \alpha_{n1}&\ldots&\alpha_{nn}\end{pmatrix}$ for real $\lambda_1,\lambda_2,\ldots,\lambda_n$ numbers, for $\forall\ j_1,\ldots,j_m$ be $\lambda_{j_1}+\ldots+\lambda_{j_m}\geq 0$. If $\forall\ p=(p_{00},p_{sj})$, $s=\overline{m+1,n},\ \ j=\overline{1,m}$, it is

$$\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_k\left(\alpha_{jk}p_{00}+\sum_{s=m+1}^{n}(-1)^{s+j}p_{sj}\alpha_{sk}\right)^2\geq 0$$

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We get the special derivative from Ω with respect to p_{00} and p_{js}

$$\frac{\partial\Omega}{\partial p_{00}}=\sum_{j=1}^{n}\sum_{k=1}^{n}2\lambda_k\alpha_{jk}\left(\alpha_{jk}p_{00}+\sum_{s=m+1}^{n}(-1)^{s+j}p_{js}\alpha_{sk}\right)$$

$$\frac{\partial\Omega}{\partial p_{ij}}=\sum_{k=1}^{n}2(-1)^{i+j}\lambda_k\alpha_{jk}\left(\alpha_{jk}p_{00}+\sum_{s=m+1}^{n}(-1)^{s+i}p_{is}\alpha_{sk}\right)$$ setting each to 0,

$$\begin{cases}\dfrac{\partial\Omega}{\partial p_{00}}=\sum_{j=1}^{n}\sum_{k=1}^{n}2\lambda_k\alpha_{jk}\left(\alpha_{jk}p_{00}+\sum_{s=m+1}^{n}(-1)^{s+j}p_{js}\alpha_{sk}\right)=0\\ \dfrac{\partial\Omega}{\partial p_{ij}}=\sum_{k=1}^{n}2(-1)^{i+j}\lambda_k\alpha_{jk}\left(\alpha_{jk}p_{00}+\sum_{s=m+1}^{n}(-1)^{s+i}p_{is}\alpha_{sk}\right)=0\end{cases}$$

we find the solution of the system $$\begin{cases}\dfrac{\partial\Omega}{\partial p_{00}}=\sum_{k=1}^{n}2\lambda_k\alpha_{1k}\left(\alpha_{1k}p_{00}+\sum_{s=m+1}^{n}(-1)^{s+1}p_{1s}\alpha_{sk}\right)=0\\ \vdots\\ \dfrac{\partial\Omega}{\partial p_{00}}=\sum_{k=1}^{n}2\lambda_k\alpha_{mk}\left(\alpha_{mk}p_{00}+\sum_{s=m+1}^{n}(-1)^{s+m}p_{ms}\alpha_{sk}\right)=0\end{cases}$$

$$\begin{cases}\dfrac{\partial\Omega}{\partial p_{1m+1}}=\sum_{k=1}^{n}2\lambda_k(-1)^{m+2}\alpha_{m+1k}\left(\alpha_{m+1k}p_{00}+\sum_{s=m+1}^{n}(-1)^{1+s}p_{1s}\alpha_{sk}\right)=0\\ \vdots\\ \dfrac{\partial\Omega}{\partial p_{1n}}=\sum_{k=1}^{n}2\lambda_k(-1)^{m+2}\alpha_{m+1k}\left(\alpha_{m+1k}p_{00}+\sum_{s=m+1}^{n}(-1)^{1+s}p_{1s}\alpha_{sk}\right)=0\end{cases}$$

$$\begin{cases}\dfrac{\partial\Omega}{\partial p_{2m+1}}=\sum_{k=1}^{n}2\lambda_k(-1)^{m+3}\alpha_{m+1k}\left(\alpha_{m+1k}p_{00}+\sum_{s=m+1}^{n}(-1)^{2+s}p_{2s}\alpha_{sk}\right)=0\\ \vdots\\ \dfrac{\partial\Omega}{\partial p_{2n}}=\sum_{k=1}^{n}2\lambda_k(-1)^{2+n}\alpha_{m+1k}\left(\alpha_{m+1k}p_{00}+\sum_{s=m+1}^{n}(-1)^{2+s}p_{2s}\alpha_{sk}\right)=0\end{cases}$$

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$$\begin{cases} \dfrac{\partial \Omega}{\partial p_{mm+1}} = \sum_{k=1}^{n} 2\lambda_k (-1)^{m+m+1} \alpha_{m+1k} \left(\alpha_{m+1k} p_{00} + \sum_{s=m+1}^{n} (-1)^{m+s} p_{ms} \alpha_{sk} \right) = 0 \\ \vdots \\ \dfrac{\partial \Omega}{\partial p_{2n}} = \sum_{k=1}^{n} 2\lambda_k (-1)^{m+n} \alpha_{m+1k} \left(\alpha_{m+1k} p_{00} + \sum_{s=m+1}^{n} (-1)^{m+s} p_{ms} \alpha_{sk} \right) = 0 \end{cases} .$$

This system has the solution of $p_{00} = p_{ij} = 0,\ i = \overline{1,m};\ j = \overline{m+1,n}$.

It suffices to show that the Hessian functional Ω becomes nonnegative around this point sufficiently small. We find the second-order derivative of the Ω function in regarding p_{00} :

$$\frac{\partial \Omega}{\partial p_{00}} = \sum_{j=1}^{m} \sum_{k=1}^{n} 2\lambda_k \alpha_{jk} \left(\alpha_{jk} p_{00} + \sum_{s=m+1}^{n} (-1)^{j+s} p_{js} \alpha_{sk} \right)$$

$$\frac{\partial^2 \Omega}{\partial p_{00}^2} = \sum_{j=1}^{m} \sum_{k=1}^{n} 2\lambda_k \alpha_{jk}^2$$

$$\frac{\partial^2 \Omega}{\partial p_{00} \partial p_{qs}} = \sum_{k=1}^{n} 2\lambda_k \alpha_{qk} \alpha_{qs}$$

We find the second-order derivative of the functional Ω with respect to p_{ij} :

$$\frac{\partial \Omega}{\partial p_{ij}} = 2\sum_{k=1}^{n} \lambda_k \alpha_{jk} (-1)^{i+j} \left(\alpha_{ik} p_{00} + \sum_{s=m+1}^{n} (-1)^{i+s} p_{is} \alpha_{sk} \right),$$

$$\frac{\partial^2 \Omega}{\partial p_{ij}^2} = 2\sum_{k=1}^{n} \lambda_k \alpha_{jk}^2 ,$$

$$\frac{\partial^2 \Omega}{\partial p_{ij} \partial p_{lq}} = 0$$

$$\frac{\partial^2 \Omega}{\partial p_{ij} \partial p_{iq}} = 2\sum_{k=1}^{n} \lambda_k \left(-1\right)^{j+q} \alpha_{jk} \alpha_{qk}$$

We see that the second-order derivatives are positively defined in the following matrix:

$$\begin{pmatrix}
\sum\limits_{j=1}^{m}\sum\limits_{k=1}^{n}\lambda_k\alpha_{jk}^2 & \sum\limits_{k=1}^{n}\lambda_k\alpha_{1k}\alpha_{km+1} & \sum\limits_{k=1}^{n}\lambda_k\alpha_{1k}\alpha_{km+2} & \ldots\sum\limits_{k=1}^{n}\lambda_k\alpha_{1k}\alpha_{kn} & \sum\limits_{k=1}^{n}\lambda_k\alpha_{mk}\alpha_{km+1} & \ldots & \sum\limits_{k=1}^{n}\lambda_k\alpha_{mk}\alpha_{kn} \\
\sum\limits_{k=1}^{n}\lambda_k\alpha_{1k}\alpha_{km+1} & \sum\limits_{k=1}^{n}\lambda_k\alpha_{1k}^2 & \left(-1\right)^{m+2}\sum\limits_{k=1}^{n}\lambda_k\alpha_{1k}\alpha_{m+1k} & \ldots & \left(-1\right)^{n+1}\sum\limits_{k=1}^{n}\lambda_k\alpha_{mk}\alpha_{nk} & \ldots & 0 \\
\sum\limits_{k=1}^{n}\lambda_k\alpha_{1k}\alpha_{km+2} & \left(-1\right)^{m+2}\sum\limits_{k=1}^{n}\lambda_k\alpha_{1k}\alpha_{m+1k} & \sum\limits_{k=1}^{n}\lambda_k\alpha_{1k}^2 & \ldots & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & & & & \\
\sum\limits_{k=1}^{n}\lambda_k\alpha_{1k}\alpha_{kn} & \left(-1\right)^{n+1}\sum\limits_{k=1}^{n}\lambda_k\alpha_{mk}\alpha_{nk} & & \ddots & \ldots & 0 & 0 \\
\sum\limits_{k=1}^{n}\lambda_k\alpha_{mk}\alpha_{km+1} & 0 & \vdots & \vdots & \sum\limits_{k=1}^{n}\lambda_k\alpha_{mk}^2 & \ldots & \left(-1\right)^{m+n+2}\sum\limits_{k=1}^{n}\lambda_k\alpha_{mk}\alpha_{nk} \\
\vdots & \vdots & 0 & 0 & \ldots & \ddots & 0 \\
\sum\limits_{k=1}^{n}\lambda_k\alpha_{mk}\alpha_{kn} & 0 & 0 & 0 & \left(-1\right)^{m+n+2}\sum\limits_{k=1}^{n}\lambda_k\alpha_{mk}\alpha_{nk} & \ldots & \sum\limits_{k=1}^{n}\lambda_k\alpha_{nk}^2
\end{pmatrix}$$

This symmetric $(m(n-m)+1)-$-dimensional square matrix is nonnegative definite. The theorem is proved.

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