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Research Article

METHODOLOGY FOR STUDYING SOME NON – INTGERABLE FUNCTIONS IN AN UNCONVENTIONAL WAY

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ABSTRACT

This article discusses a method for calculating functions belonging to the class of functions that are not intgerable in a standard way using the Feynman method, which allows one to obtain an exact analytical solution. This article shows a method for calculating some rather complex integrals that cannot be integrated in the standard way.

KEYWORDS

Definite integral, improper integrals, Feynman's trick, Dirichlet integral, differentiation of an integral with respect to a parameter.

INTRODUCTION

Some integrals, which belong to the class of functions that are not intgerable by standard methods, can be calculated using a method created by Nobel Prize winner (1965) Richard Feynman. Richard developed an integration method called the Feynman trick. He has achieved achievements in the field of theoretical physics, the development of a method of integration along trajectories from quantum mechanics, and the reformation of teaching methods in higher educational institutions.

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Integration is associated with important methods of analysis and study of numerical functions – averages, limits, infinitesimals, differentials, derivatives, and so on, and therefore without understanding and studying these concepts, the study of functions is impossible. To find the value of the integral, scientists such as Richard Feynman found unconventional methods. Integrating functions is a mathematical art. It is interesting to calculate them, especially when non-standard methods are used in the solution.

Consider the integral

$I(p)=\int_{a(p)}^{b(p)} f(x,p)dx$ where p – is the integral parameter, x – is the integration variable.

$$I'(p)=\frac{\partial}{\partial p}\int_a^b f(x,p)dx$$

Consider the improper integral (Dirichlet integral)

$$\int_{-\infty}^{\infty}\frac{\sin ax}{x}dx \qquad (1) \qquad a>0$$

The integrand is even and therefore

$$\int_{-\infty}^{\infty}\frac{\sin ax}{x}dx=2\int_0^{\infty}\frac{\sin ax}{x}dx \qquad (2)$$

To calculate the right side of equality (2), we find the function

$$\frac{1}{x} \quad \text{in the form} \quad \int_0^{\infty}e^{-xt}dt=\frac{1}{x}$$

where $a>0,\ a=const$. Then form equality (2) we obtain

$$2\int_0^{\infty}\frac{\sin ax}{x}dx=2\int_0^{\infty}\sin ax\left(\int_0^{\infty}e^{-tx}dt\right)dx=2\int_0^{\infty}\sin axdx\cdot\int_0^{\infty}e^{-tx}dt=2\int_0^{\infty}dt\int_0^{\infty}\sin axe^{-tx}dx$$

To calculate $\int_0^{\infty}\sin axe^{-tx}dx$, we use Euler's formula

$$\sin ax=\frac{e^{iax}-e^{-iax}}{2i} \qquad (3)$$

Then $$\int_0^{\infty}\sin axe^{-xt}dx=2\int_0^{\infty}dt\int_0^{\infty}\frac{e^{iax}-e^{-iax}}{2i}e^{-tx}dx=\frac{1}{i}\int_0^{\infty}\frac{2ia}{t^2+a^2}dt=2a\int_0^{\infty}\frac{dt}{t^2+a^2}=$$

$$=2a\cdot\frac{1}{a}arctg\frac{t}{a}\Big|_0^{\infty}=2\cdot\frac{\pi}{2}=\pi$$

$$2\int_0^{\infty}\frac{\sin ax}{x}dx=\pi;\qquad \int_0^{\infty}\frac{\sin ax}{x}dx=\frac{\pi}{2}$$

Now let's calculate the same integral (1) using Feynman's method. Thus, we will demonstrate the power of Feynman's method.

$$\int_0^{\infty}\frac{\sin ax}{x}dx$$

Solution. Let function $I(p)$ be defined by the formula

$$I(p)=\int_{-\infty}^{\infty}e^{-px}\frac{\sin ax}{x}dx,\qquad p>0,\text{ where } a>0$$

Differentiating with respect to the parameter p, where x is a fictitious integration variable, we obtain

$$\frac{dI}{dp}=\int_{-\infty}^{\infty}\frac{\partial}{\partial p}\left(e^{-px}\cdot\frac{\sin ax}{x}\right)dx=\int_{-\infty}^{\infty}-xe^{-px}\frac{\sin ax}{x}dx=\int_{-\infty}^{\infty}-e^{-px}\sin axdx$$

By integrating the last integral twice by parts, it is not difficult to show that $\frac{dI}{dp}=\frac{-a}{a^2+p^2}$.

After integration we get $I(p)=-arctg\frac{p}{a}+C$, where C is an arbitrary integration constant. We can calculate C by noting that $I(\infty)=0$ in the original integral definition of $g(p)$, because the factor e^{-xp} in the integral tends to zero everywhere at $p\to\infty$ (because $x\geq 0$ throughout the integration interval). So $0=C-arctg(\pm\infty)$, where we use a plus sign if $a>0$, and a minus sign if $a<0$. So, $C=\pm\frac{\pi}{2}$ and we have

$$I(p)=\pm\frac{\pi}{2}-arctg\frac{p}{a}$$

For $p=0$ (and therefore $arctg\frac{p}{a}=0$) gives us the following remarkable result, called the discontinuous Dirichlet integral. Ultimately we see that in both cases the result is the same

$$\int_0^{\infty}\frac{\sin ax}{x}dx=\begin{cases}\frac{\pi}{2}, & if\quad a>0\\ 0, & if\quad a=0\\ -\frac{\pi}{2}, & if\quad a<0\end{cases}$$

Example – 2. Calculate the integral

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$$\int_0^{\pi} \ln\left(\frac{2+\sqrt{3}\cos x}{2}\right)\cdot\frac{dx}{\cos x} \qquad (4)$$

Solution. The integral function is continuous at all points of the segment $[0;\ \pi]$, except for point $x=\frac{\pi}{2}$. At this point we have uncertainty of the form $\left(\frac{0}{0}\right)$, that is, when

$$x=\frac{\pi}{2}\Rightarrow\begin{cases}\cos\frac{\pi}{2}=0\\ \ln\left(\dfrac{2+\sqrt{3}\cos\frac{\pi}{2}}{2}\right)=\ln 1=0\end{cases}$$

Let us transform the integrand expression as follows:

$$\int_0^{\pi} \ln\left(\frac{2+\sqrt{3}\cos x}{2}\right)\cdot\frac{dx}{\cos x}=\int_0^{\pi} \ln\left(1+\frac{\sqrt{3}}{2}\cos x\right)\cdot\frac{dx}{\cos x} \quad (5)$$

In the last integral we denote coefficient $\frac{\sqrt{3}}{2}$ as parameter p. For different values of the parameter p, different values of the integral will be obtained. Let's write the integral in general form.

$$I(p)=\int_0^{\pi} \ln(1+p\cos x)\cdot\frac{dx}{\cos x} \qquad (6)$$

Thus, we can say that this integral defines a function with respect to the variable *p*. in equality (6), it makes sense to consider *p* only at $-1\le p\le 1$. Only for such values of *p* is the argument of the natural logarithm greater than zero at all points of the integration segment. This means that the original integral (5) will be a special case of the integral (6) at $p=\frac{\sqrt{3}}{2}$. Now we take the derivative with respect to the parameter *p*:

$$I'(p)=\int_0^{\pi}\frac{\partial}{\partial p}\left(\frac{\ln(1+p\cos x)}{\cos x}\right)\cdot dx$$

Here the limits of integration do not depend on the variable *p.* The partial derivative of the integrand with respect to the variable *p* will be a continuous function. Therefore, we can enter the differential under the integral sign and calculate it. When integrating.

$$I'(p)=\int_0^{\pi}\frac{\partial}{\partial p}\left(\frac{\ln(1+p\cos x)}{\cos x}\right)\cdot dx=\int_0^{\pi}\frac{\cos x dx}{(1+p\cos x)\cos x}=\int_0^{\pi}\frac{dx}{1+p\cos x}$$

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We use the universal integration method. Let $t = tg\frac{x}{2}$, then $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$. When replacing, you need to change the limits integration in the integral. If $x \to 0$ to $t \to 0$, if $x \to \pi$, to $t \to \infty$.

Then

$$I'(p) = \int_0^{\pi} \frac{dx}{\cos^2\frac{x}{2}\left(1+p-(1-p)tg^2\frac{x}{2}\right)} = \int_0^{\infty} \frac{2dt}{1+p+(1-p)t^2} = \frac{1}{1-p}\int_0^{\infty} \frac{2dt}{\left(\sqrt{\frac{1+p}{1-p}}\right)^2 + t^2} =$$

$$\frac{2}{1-p}\sqrt{\frac{1-p}{1+p}}\, arctg\frac{t}{\sqrt{\frac{1+p}{1-p}}}\Bigg|_0^{\infty} = \frac{2}{\sqrt{1-p^2}}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{1-p^2}}.\ \text{Thus } I'(p) = \frac{\pi}{\sqrt{1-p^2}}$$

$$I(p) = \pi \arcsin p + C \qquad (7)$$

From here we find *C*. $I(0) = 0$ means $C = 0$. The problem statement was that it was necessary to calculate a special case of integral (6) for $p = \frac{\sqrt{3}}{2}$. If $I(p) = \pi \arcsin p$. Then

$$I\left(\frac{\sqrt{3}}{2}\right) = \int_0^{\pi} \ln\left(1+\frac{\sqrt{3}}{2}\cos x\right)\cdot\frac{dx}{\cos x} = \pi\arcsin\frac{\sqrt{3}}{2} = \pi\cdot\frac{\pi}{3} = \frac{\pi^2}{3}$$

Example – 3. Calculate the integral

$$\int_0^1 \frac{arctg\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}\cdot dx \qquad (8)$$

Solution. Let us simplify the integrand expressions by making the following substitution: $x = \sin t$ and the boundaries will have the following form

$x = 0 \to t = 0,\ x = 1 \to t = \frac{\pi}{2}$ and we see that in the first quarter $\sin t > 0,\ \cos t > 0$. Then

$$I = \int_0^1 \frac{arctg\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}dx = \int_0^{\frac{\pi}{2}} \frac{arctg(\cos t)dt}{\cos t} \qquad (9)$$

Now let's use Feynman's method as follows

$$I'(p) = \int_0^{\frac{\pi}{2}} \frac{arctg(p\cos t)dt}{\cos t} \qquad (10)$$

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From (9) it is clear that the parameter $p=1$. In this case, our is to calculate the integral $p=1$, if we know that $I(0)=0$.

Differentiating with respect to the parameter p we get:

$$I'(p)=\frac{d}{dp}\int_0^{\frac{\pi}{2}}\frac{arctg(p\cos t)dt}{\cos t}=\int_0^{\frac{\pi}{2}}\frac{\partial}{\partial p}\left(\frac{arctg(p\cos t)}{\cos t}\right)=\int_0^{\frac{\pi}{2}}\frac{\cos t dt}{(1+p^2\cos^2 t)\cos t}=\int_0^{\frac{\pi}{2}}\frac{dt}{1+p^2\cos^2 t}=$$

$$=\int_0^{\frac{\pi}{2}}\frac{dt}{\sin^2 t+\cos^2 t+p^2\cos^2 t}=\int_0^{\frac{\pi}{2}}\frac{dt}{\sin^2 t+(1+p^2)\cos^2 t}=\int_0^{\frac{\pi}{2}}\frac{dt}{\cos^2 t(1+p^2+tg^2 t)}=$$

$$=\int_0^{\frac{\pi}{2}}\frac{d(tgt)}{1+p^2+tg^2 t}=\int_0^{\frac{\pi}{2}}\frac{d(tgt)}{\left(\sqrt{1+p^2}\right)^2+tg^2 t}=\frac{1}{\sqrt{1+p^2}}arctg\left(\frac{tgt}{\sqrt{1+p^2}}\right)\Bigg|_0^{\frac{\pi}{2}}=\frac{\pi}{2}\cdot\frac{1}{\sqrt{1+p^2}}=\frac{\pi}{2\sqrt{1+p^2}}$$

Means

$$I'(p)=\frac{\pi}{2\sqrt{1+p^2}},\quad I(p)=\frac{\pi}{2}\ln\left(p+\sqrt{1+p^2}\right)+C$$

$$I(0)=0\Rightarrow C=0\qquad I(p)=\frac{\pi}{2}\ln\left(p+\sqrt{1+p^2}\right)$$

$$I(1)=I=\int_0^1\frac{arctg\sqrt{1-x^2}}{\sqrt{1-x^2}}dx=\frac{\pi}{2}\ln\left(1+\sqrt{2}\right)$$

Example – 4. Calculate the integral

$$\int_{-\infty}^{\infty}\frac{\cos(3x)}{x^2+4}\cdot dx$$

Solution. $\int_{-\infty}^{\infty}\frac{\cos(3x)dx}{x^2+4}=2\int_0^{\infty}\frac{\cos(3x)}{x^2+4}dx$

$$I(p)=2\int_0^{\infty}\frac{\cos(px)}{x^2+4}dx$$

$$I(0)=\frac{\pi}{2}$$

$$u=px$$

$$x=\frac{u}{p}$$

$$dx=\frac{du}{p}$$

$$I'(p)=2\int_0^\infty \frac{\partial}{\partial p}\left(\frac{\cos(px)}{x^2+4}\right)dx=-2\int_0^\infty \frac{x\sin(px)}{x^2+4}dx=-2\int_0^\infty \frac{x^2\sin(px)dx}{x(x^2+4)}=$$

$$=-2\int_0^\infty \frac{(x^2+4-4)\sin(px)}{x(x^2+4)}dx=-2\int_0^\infty \frac{\sin(px)}{x}dx+8\int_0^\infty \frac{\sin(px)}{x(x^2+4)}dx=$$

$$=-2\int_0^\infty \frac{\sin u}{u}du+8\int_0^\infty \frac{\sin(px)}{x(x^2+4)}dx=-2\cdot\frac{\pi}{2}+8\int_0^\infty \frac{\sin(px)}{x(x^2+4)}dx=-\pi+8\int_0^\infty \frac{\sin(px)}{x(x^2+4)}dx$$

Means

$$I'(p)=-\pi+8\int_0^\infty \frac{\sin(px)}{x(x^2+4)}dx$$

$$I''(p)=0+8\int_0^\infty \frac{\partial}{\partial p}\left(\frac{\sin(px)}{x(x^2+4)}\right)dx=8\int_0^\infty \frac{x\cos(px)}{x(x^2+4)}dx=+8\int_0^\infty \frac{\cos(px)}{x^2+4}dx=4I(p)$$

We have obtained a homogeneous differential equation of the second order.

$I''(p)=4I(p)$

$I''(p)-4I(p)=0$

Make up a characteristic equation

$k^2-4=0$

$k_1=2,\ \ k_2=-2$

General solution equation is equal

$$I(p)=c_1e^{2p}+c_2e^{-2p}$$

$$I(0)=c_1+c_2=\frac{\pi}{2}$$

$$I'(p)=2c_1e^{2p}-2c_2e^{-2p}$$

$$I'(0)=2c_1-2c_2=-\pi$$

$$\begin{cases} c_1+c_2=\dfrac{\pi}{2} \\ 2c_1-2c_2=-\pi \end{cases} \qquad c_1=0,\ \ c_2=\frac{\pi}{2}$$

$$I(p)=\frac{\pi}{2}e^{-2p}$$

$$I(3)=\int_{-\infty}^{\infty}\frac{\cos(3x)}{x^2+4}dx=\frac{\pi}{2}e^{-6}=\frac{\pi}{2e^6}$$

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Conclusions

Integrating functions is a mathematical art. It is interesting to calculate them, especially when non-standard methods are used in the solution. The article uses Feynman's method to calculate some rather complex integrals that cannot be integrated in the standard way.

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